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Decomposition of Jordan automorphisms of strictly triangular matrix algebra over local rings

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Abstract

Let $N_{n+1}(R)$ be the algebra of all strictly upper triangular $n+1$ by $n+1$ matrices over a 2-torsionfree commutative local ring R with identity. In this paper, we prove that any Jordan automorphism of $N_{n+1}(R)$ can be uniquely written as a product of a graph automorphism, a diagonal automorphism, an inner automorphism and a central automorphism for $n \geqslant 3$. In the cases $n = 1, 2$, we also give a decomposition for any Jordan automorphism of $N_{n+1}(R)$ $(1 \leqslant n \leqslant 2)$.

AMS classification: 17C30; 17C50; 13C10

Keywords: Jordan automorphism; Strictly triangular matrix algebra; Local ring

1. Introduction

Let R be a commutative ring with identity 1 and Γ an algebra over R. Recall that a bijective R-linear map $\varphi : \Gamma \to \Gamma$ is called a *Jordan automorphism* if $\varphi(AB + BA) = \varphi(A)\varphi(B) + \varphi(B)\varphi(A)$ for all $A, B \in \Gamma$. In the past half-century many

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authors studied Jordan isomorphism of associative algebras (see [1–4,7,10]). Obviously, automorphism and anti-automorphism both are examples of Jordan automorphisms. Recently, Beidark et al. [3] gave an example of Jordan automorphisms which is neither an automorphism nor an anti-automorphism.

The algebra $T_n(R)$ of all triangular matrices over R is an interesting topic for many researchers. Many papers are concerned with the study of automorphisms and Lie automorphisms of $T_n(R)$ (see [3,5,8–10]). Two years ago Cao [6] studied the automorphisms of the nilpotent Lie algebra which consists of all strictly upper triangular $(n+1)\times(n+1)$ matrices over a local ring with 2 as a unit or an integral domain of characteristic not 2, and gave a decomposition for the above automorphisms. This work encourages us to consider an analogue problem concerning Jordan automorphism of $N_{n+1}(R)$, the algebra consisting of all strictly upper triangular $(n+1)\times(n+1)$ matrices over R, where R is a 2-torsionfree local ring.

In this article we prove that any Jordan automorphism φ of $N_{n+1}(R)$ can be uniquely expressed as $\varphi=\omega\lambda_d\zeta_c\theta$, where ω, λ_d, ζ_c and θ are graph, diagonal, central and inner automorphisms respectively for $n\geqslant 3$ and R is a 2-torsionfree local ring. In the rest cases, we also show that any Jordan automorphism of $N_2(R)$ $(n=1)$ is a diagonal automorphism and that any Jordan automorphism of $N_3(R)$ $(n=2)$ can be written as a product of graph, diagonal and inner automorphisms.

2. Preliminaries

Throughout this paper, $M_{n+1}(R)$ denotes the R-algebra of all $(n+1)\times(n+1)$ matrices over a 2-torsionfree local ring R (the identity of R is denoted by 1). Let e denote the identity matrix of $M_{n+1}(R)$ and e_{ij} the matrix with 1 at the position (i,j) and zeros elsewhere. Jordan multiplication can be defined on $M_{n+1}(R)$ by $x\circ y=xy+yx$ for all $x,y\in M_{n+1}(R)$. The subalgebra of $M_{n+1}(R)$ consisting of all strictly upper triangular matrices is denoted by $N_{n+1}(R)$. The matrix set $\{e_{i,i+k}\mid i=1,\dots,n-k+1,k=1,\dots,n\}$ is a basis of $N_{n+1}(R)$. For any $x\in N_{n+1}(R)$, we can write $x=\sum_{k=1}^{n}\sum_{i=1}^{n-k+1}a_{i,i+k}e_{i,i+k}$ for some $a_{i,i+k}\in R$. Let $\mathbf{n}_1=N_{n+1}(R)$. We have $\mathbf{n}_2=\mathbf{n}_1\circ\mathbf{n}_1$, $\mathbf{n}_j=\mathbf{n}_1\circ\mathbf{n}_{j-1}$, where $\mathbf{n}_j=\sum_{m=j}^{n}\sum_{i=1}^{n-m+1}Re_{i,i+m}$ and $j=2,3,\dots,n$. $\mathbf{n}_n=Re_{1,n+1}$ is the center of $N_{n+1}(R)$. An element in $\mathbf{n}_k$ is often denoted by t_k. It is easy to check that for any $t_m\in\mathbf{n}_m$, $t_l\in\mathbf{n}_l$, t_mt_l and $t_m\circ t_l$ both lie in $\mathbf{n}_{m+l}$ for $m+l\leqslant n$ or are equal to zero for $m+l>n$. Let $\mathrm{Aut}(\mathbf{n}_1)$ denote the Jordan automorphism group of $N_{n+1}(R)$. For any $\varphi\in\mathrm{Aut}(\mathbf{n}_1)$, we have $\varphi(\mathbf{n}_1)=\mathbf{n}_1$, $\varphi(\mathbf{n}_2)=\varphi(\mathbf{n}_1)\circ\varphi(\mathbf{n}_1)=\mathbf{n}_1\circ\mathbf{n}_1=\mathbf{n}_2,\dots,\varphi(\mathbf{n}_j)=\mathbf{n}_j,\ \ j=2,3,\dots,n.$ Therefore $\varphi(\mathbf{n}_j\backslash\mathbf{n}_{j+1})=\mathbf{n}_j\backslash\mathbf{n}_{j+1}$, $j=1,2,\dots,n-1$.

Let M be the unique maximal ideal of R, and $\bar{R}=R/M$ the residue field. The natural homomorphism $\pi:R\to R/M$ induces a homomorphism (we still denote it by π) $\pi:N_{n+1}(R)\to N_{n+1}(\bar{R})$. So every Jordan automorphism φ may induce a Jordan automorphism $\bar{\varphi}$ of $N_{n+1}(\bar{R})$. Using the above fact and that $\mathbf{n}_n=Re_{1,n+1}$ is the center of $N_{n+1}(R)$, we may show that $\varphi(e_{1,n+1}) = ae_{1,n+1}$ where $a \in R^*$, the set of all invertible elements in R. Otherwise, if $a \notin R^*$ $(a \in M)$, then $\bar{\varphi}(\bar{e}_{1,n+1}) = 0$ on $N_{n+1}(\bar{R})$, where $\bar{e}_{1,n+1}$ is the image of $e_{1,n+1}$ in $N_{n+1}(\bar{R})$, which is impossible.

Lemma 2.1. *Let φ be a R-module automorphism of $N_{n+1}(R)$. The following two statements are equivalent*:

(i) *φ is in* $\mathrm{Aut}(\mathbf{n}_1)$;
(ii) $\varphi(e_{i,i+k}) = \varphi(e_{i,i+l}) \circ \varphi(e_{i+l,i+k})$ *for all* $1 \leqslant l < k$ *and* $\varphi(e_{i,i+j}) \circ \varphi(e_{l,l+k}) = 0$ *for all* $i + j < l$.

Proof. It is obvious that (i) $\Rightarrow$ (ii). Let us show that (ii) $\Rightarrow$ (i). For any $e_{i,i+k}, e_{j,j+l} \in N_{n+1}(R)$, we have $e_{i,i+k} \circ e_{j,j+l} = e_{i,i+k}e_{j,j+l} = e_{j,j+l} \circ e_{i,i+k}$ for $i + k \leqslant j$. If $i + k = j$ or $j + l = i$, then $\varphi(e_{i,i+k} \circ e_{j,j+l}) = \varphi(e_{i,i+k}) \circ \varphi(e_{j,j+l})$. If $i + k \neq j$ and $j + l \neq i$, then $e_{i,i+k} \circ e_{j,j+l} = 0$ and we have $\varphi(e_{i,i+k} \circ e_{j,j+l}) = 0 = \varphi(e_{i,i+k}) \circ \varphi(e_{j,j+l})$. Thus for any $e_{i,i+k}, e_{j,j+l} \in N_{n+1}(R)$, $\varphi(e_{i,i+k} \circ e_{j,j+l}) = \varphi(e_{i,i+k}) \circ \varphi(e_{j,j+l})$ holds, so that for any $x, y \in N_{n+1}(R)$, we have $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$. Hence $\varphi \in \mathrm{Aut}(\mathbf{n}_1)$. $\square$

Lemma 2.1 implies that the set $\{\varphi(e_{i,i+1}) \mid i = 1, \ldots, n\}$ generates $N_{n+1}(R)$. So we will investigate $\varphi(e_{i,i+1})$ $(i = 1, \ldots, n)$ and we write $\varphi(e_{i,i+1})$ as

$$\varphi(e_{i,i+1}) = \sum_{k=1}^{n} a_{k,k+1}^{(i)} e_{k,k+1} \bmod \mathbf{n}_2. \tag{2.1}$$

It is known that when R is a 2-torsionfree, a Jordan automorphism φ of $N_{n+1}(R)$ is also a semi-automorphism of $N_{n+1}(R)$, that is, $\varphi(x^2) = [\varphi(x)]^2$ and $\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x)$ for any $x, y \in N_{n+1}(R)$.

Lemma 2.2. *Suppose that R is 2-torsionfree and $n \geqslant 3$. Then for any $e_{i,i+k}, x \in N_{n+1}(R)$ and any $\varphi \in \mathrm{Aut}(\mathbf{n}_1)$, we have $[\varphi(e_{i,i+k})]^2 = 0$ and $\varphi(e_{i,i+k})x\varphi(e_{i,i+k}) = 0$.*

Proof. We only need to point out that $(e_{i,i+k})^2 = 0$ and $e_{i,i+k}xe_{i,i+k} = 0$ for any $e_{i,i+k}, x \in N_{n+1}(R)$, and the latter equality implies that $e_{i,i+k}\varphi^{-1}(x)e_{i,i+k} = 0$. $\square$

For convenience, we write $e_{i,i+k}$ as e_{ij} where $j > i$ in some cases.

Lemma 2.3. *Let $\varphi \in \mathrm{Aut}(\mathbf{n}_1)$. Then*

(i) $\varphi(e_{12}) = a_{12}^{(1)} e_{12} + t_2$ *where* $a_{12}^{(1)} \in R^*$ *and* $t_2 \in \mathbf{n}_2$ *or* $\varphi(e_{12}) = a_{n,n+1}^{(1)} e_{n,n+1} + t_2$ *where* $a_{n,n+1}^{(1)} \in R^*$ *and* $t_2 \in \mathbf{n}_2$.

(ii) *If* $\varphi(e_{12}) = a_{12}^{(1)} e_{12} + t_2$ *where* $a_{12}^{(1)} \in R^*$ *and* $t_2 \in \mathbf{n}_2$, *then* $\varphi(e_{i,i+1}) = a_{i,i+1}^{(i)} e_{i,i+1} + t_2$, *where* $a_{i,i+1}^{(i)} \in R^*$.

Proof. (i) By (2.1)

$$\varphi(e_{12}) = \sum_{k=1}^{n} a_{k,k+1}^{(1)} e_{k,k+1} + t_2.$$

Since $e_{2,n+1} \in \mathbf{n}_{n-1} \backslash \mathbf{n}_n$, we have $\varphi(e_{2,n+1}) \in \mathbf{n}_{n-1} \backslash \mathbf{n}_n$. Assume

$$\varphi(e_{2,n+1}) = ae_{1n} + be_{2,n+1} + ce_{1,n+1}.$$

Then

$$\varphi(e_{1,n+1}) = \varphi(e_{12}) \circ \varphi(e_{2,n+1}) = \left(a_{12}^{(1)} b + aa_{n,n+1}^{(1)}\right) e_{1,n+1}.$$

We have stated that $a_{12}^{(1)} b + aa_{n,n+1}^{(1)}$ must be in R^*, so either $a_{12}^{(1)}$ or $a_{n,n+1}^{(1)}$ is in R^* since R is a local ring. Without loss of generality, we assume $a_{12}^{(1)} \in R^*$. By Lemma 2.2, it is not difficult to verify that

$$\varphi(e_{12}) e_{2k} \varphi(e_{12}) = a_{12}^{(1)} a_{k,k+1}^{(1)} e_{1,k+1} + t_{k+1} = 0,$$

so we have $a_{12}^{(1)} a_{k,k+1}^{(1)} = 0$ for $3 \leqslant k \leqslant n$. From $[\varphi(e_{12})]^2 = 0$ we have $a_{12}^{(1)} a_{23}^{(1)} = 0$. Since $a_{12}^{(1)} \in R^*$, all $a_{k,k+1}^{(1)} = 0$ for $2 \leqslant k \leqslant n$. Similarly if $a_{n,n+1}^{(1)} \in R^*$ we can show that $a_{k,k+1}^{(1)} = 0$ for $1 \leqslant k \leqslant n-1$. Then (i) is completed.

(ii) Write $\varphi(e_{i,i+1})$ as

$$\varphi(e_{i,i+1}) = \sum_{k=1}^{n} a_{k,k+1}^{(i)} e_{k,k+1} + t_2.$$

By assumption, we may prove the conclusion by induction on k. Assume that $a_{k,k+1}^{(k)} \in R^*$ and $\varphi(e_{k,k+1}) = a_{k,k+1}^{(k)} e_{k,k+1} + t_2$ for $2 \leqslant k \leqslant m-1$ hold. Since $e_{1,m} = e_{12} \circ e_{23} \circ \cdots \circ e_{m-1,m}, \varphi(e_{1,m}) = a_{12}^{(1)} \cdots a_{m-1,m}^{(m-1)} e_{1,m} + t_m$ holds. Further, since $e_{1,m+1} \in \mathbf{n}_m \backslash \mathbf{n}_{m+1}$, we have

$$\begin{aligned}\varphi(e_{1,m+1}) &= \varphi(e_{1,m}) \circ \varphi(e_{m,m+1}) \\ &= a_{12}^{(1)} \cdots a_{m-1,m}^{(m-1)} a_{m,m+1}^{(m)} e_{1,m+1} + t_{m+1} \in \mathbf{n}_m \backslash \mathbf{n}_{m+1}.\end{aligned}$$

If $a_{m,m+1}^{(m)} \notin R^*$, modularizing the maximal ideal M of R, we have $\bar{\varphi}(\bar{e}_{1,m+1}) = \bar{t}_{m+1} \in \mathbf{n}_{m+1}$ on $N_{n+1}(\bar{R})$, this is impossible. So $a_{m,m+1}^{(m)} \in R^*$.

We have proved that all $a_{k,k+1}^{(k)} \in R^*$ for $1 \leqslant k \leqslant n$. Now we distinguish the following two cases:

(a) $1 \leqslant n \leqslant 4$

When $n = 1$, $\varphi(e_{12}) = a_{12}^{(1)} e_{12}$, it is trivial.

When $n = 2$, by $[\varphi(e_{23})]^2 = 0$ we have $a_{12}^{(2)} a_{23}^{(2)} = 0$, that is, $a_{12}^{(2)} = 0$.

When $n = 3$, by $[\varphi(e_{23})]^2 = 0$ we have $a_{12}^{(2)} a_{23}^{(2)} = 0$ and $a_{23}^{(2)} a_{34}^{(2)} = 0$. Since $a_{23}^{(2)} \in R^*$, $a_{12}^{(2)} = a_{34}^{(2)} = 0$. So $\varphi(e_{23}) = a_{23}^{(2)} e_{23} + t_2$. For $\varphi(e_{34})$, from $[\varphi(e_{34})]^2 = 0$ and $\varphi(e_{34}) e_{23} \varphi(e_{34}) = 0$ we obtain that $a_{12}^{(3)} = a_{23}^{(3)} = 0$. That is, $\varphi(e_{34}) = a_{34}^{(3)} e_{34} + t_2$.

When $n = 4$, from $\varphi(e_{23}) e_{34} \varphi(e_{23}) = 0$, $[\varphi(e_{23})]^2 = 0$, we have $a_{23}^{(2)} a_{45}^{(2)} = 0$, $a_{12}^{(2)} a_{23}^{(2)} = 0$ and $a_{23}^{(2)} a_{34}^{(2)} = 0$. Since $a_{23}^{(2)} \in R^*$, $a_{45}^{(2)} = a_{12}^{(2)} = a_{34}^{(2)} = 0$. That is, $\varphi(e_{23}) = a_{23}^{(2)} e_{23} + t_2$. Similarly, we may show that $\varphi(e_{34})$ and $\varphi(e_{45})$ have the wanted form.

(b) $n \geqslant 5$

(i) For $\varphi(e_{23})$, from $[\varphi(e_{23})]^2 = 0$, we have $a_{12}^{(2)} = a_{34}^{(2)} = 0$ (note that $a_{23}^{(2)} \in R^*$). From $\varphi(e_{23}) e_{3k} \varphi(e_{23}) = 0$ $(4 \leqslant k \leqslant n)$, we have $a_{k,k+1}^{(2)} = 0$ $(4 \leqslant k \leqslant n)$. That is, $\varphi(e_{23}) = a_{23}^{(2)} e_{23} + t_2$.

(ii) For $\varphi(e_{i,i+1})$, where $3 \leqslant i \leqslant n-2$, by the following equality:

$$\varphi(e_{i,i+1}) e_{mk} \varphi(e_{i,i+1}) = a_{m-1,m}^{(i)} a_{k,k+1}^{(i)} e_{m-1,k+1} + t_{k-m+3} = 0,$$

we have $a_{i,i+1}^{(i)} a_{j,j+1}^{(i)} = 0$ for $1 \leqslant j \leqslant i-2$ and $i+2 \leqslant j \leqslant n$. From $[\varphi(e_{i,i+1})]^2 = 0$ we obtain that $a_{i,i+1}^{(i)} a_{j,j+1}^{(i)}$=0 for $j = i-1$ and $i+1$. That means $a_{j,j+1}^{(i)} = 0$ when $j \neq i$.

(iii) For $\varphi(e_{n-1,n})$ and $\varphi(e_{n,n+1})$, using $\varphi(e_{n-1,n}) e_{m,n-1} \varphi(e_{n-1,n}) = 0$, $[\varphi(e_{n-1,n})]^2 = 0$ and also $\varphi(e_{n,n+1}) e_{mn} \varphi(e_{n,n+1}) = 0$, $[\varphi(e_{n,n+1})]^2 = 0$, we may show that $\varphi(e_{n-1,n})$ and $\varphi(e_{n,n+1})$ have the wanted form. $\square$

Now let us introduce several types of Jordan automorphism.

(i) Let $e_0 = \sum_{i=1}^{n+1} e_{i,n-i+2}$. The map $g : x \mapsto (e_0 x e_0)^t$ (t denotes the transpose of matrix) is a Jordan automorphism of $N_{n+1}(R)$. We call g a *graph automorphism* [5]. In general, graph automorphism of $N_{n+1}(R)$ is not a R-algebra automorphism, for example, $g(e_{12}e_{23}) \neq g(e_{12})g(e_{23})$. In some special case, the identity automorphism can be considered as a graph automorphism. It is easy to check that $g^{-1} = g$ and that a graph automorphism on the basis of $N_{n+1}(R)$ acts as $e_{i,i+k} \mapsto e_{n-i-k+2,n-i+2}$. The subgroup of Aut($\mathbf{n}_1$) generated by g is denoted by $\mathscr{G}$. The element in $\mathscr{G}$ is denoted by ω.

(ii) Let $d = \sum_{i=1}^{n+1} d_i e_{ii}$ where $d_i \in R^*$, $i = 1, 2, \ldots, n+1$. The map λ_d: $x \mapsto dxd^{-1}$ is called a *diagonal automorphism* which is a R-algebra automorphism of $N_{n+1}(R)$. It is obvious that $\lambda_d^{-1} = \lambda_{d^{-1}}$. The set of all diagonal automorphisms of $N_{n+1}(R)$ is a subgroup of Aut($\mathbf{n}_1$), which is denoted by $\mathscr{D}$.

(iii) Let $\sigma : \mathbf{n}_1 \to R$ be a linear map such that $\sigma(x) = 0$ for any $x \in \mathbf{n}_2$. The map $\zeta_\sigma : x \mapsto x + \sigma(x) e_{1,n+1}$ is called a *central automorphism* which is a R-algebra automorphism of $N_{n+1}(R)$. When $n \geqslant 2$ the operation of a central automorphism on the basis of $N_{n+1}(R)$ is $\zeta_c : e_{i,i+1} \mapsto e_{i,i+1} + c_i e_{1,n+1}$ for $1 \leqslant i \leqslant n$ and $e_{ij} \mapsto e_{ij}$ $(i < j)$ for other cases. Hence it uniquely determines an n-tuple $c = (c_1, \ldots, c_n) \in R^n$. Conversely, any $c = (c_1, \ldots, c_n) \in R^n$ determines a central automorphism of $\mathbf{n}_1$. In a special case $c = (0, c_2, \ldots, c_{n-1}, 0)$ we write $c = (c_2, \ldots, c_{n-1})$ for $c = (0, c_2, \ldots, c_{n-1}, 0)$. Furthermore $\zeta_c^{-1} = \zeta_{-c}$. When $n > 2$, the set of all proper central automorphisms of $\mathbf{n}_1$ is a subgroup of $\mathrm{Aut}(\mathbf{n}_1)$, which is denoted by $\mathscr{C}$.

(iv) For any $x \in N_{n+1}(R)$, let $r = e + x$. The map $\theta_r : y \mapsto ryr^{-1}$ is called an *inner automorphism* which is a R-algebra automorphism of $N_{n+1}(R)$. If $r = r_{ij}(a) = e + ae_{ij}$ $(i < j)$ with some $a \in R$, then $\theta_{r_{ij}(a)}$ is called a "simple" form. By $[r_{ij}(a)]^{-1} = r_{ij}(-a)$, we know that $\theta_{r_{ji}(a)}(e_{i,i+1}) = e_{i,i+1} + ae_{j,i+1}$ $(j < i)$, $\theta_{r_{i+1,j}(a)}(e_{i,i+1}) = e_{i,i+1} - ae_{ij}$ $(i + 1 < j)$ and $\theta_{r_{ij}(a)}(e_{k,k+1}) = e_{k,k+1}$ ($j \neq k$ and $i \neq k + 1$, where $i < j$). Note that $\theta^{-1}_{r_{ij}(a)} = \theta_{r_{ij}(-a)}$. Obviously, the set of all inner automorphisms of $\mathbf{n}_1$ is a normal subgroup of $\mathrm{Aut}(\mathbf{n}_1)$, which is denoted by $\mathscr{I}$.

Lemma 2.4. *Let φ be in $\mathrm{Aut}(\mathbf{n}_1)$. There exist a graph automorphism ω and a diagonal automorphism λ_d such that $\lambda_d\omega\varphi(e_{i,i+1}) = e_{i,i+1} + t_2$, $i = 1, 2, \ldots, n$.*

Proof. If $\varphi(e_{12}) = a_{12}^{(1)}e_{12} + t_2$ where $a_{12}^{(1)} \in R^*$, then $\varphi(e_{i,i+1}) = a_{i,i+1}^{(i)}e_{i,i+1} + t_2$, $i = 1, 2, \ldots, n$, by Lemma 2.3. In this case $\omega = 1$. Let λ_d be a diagonal automorphism such that $e_{i,i+1} \mapsto (a_{i,i+1}^{(i)})^{-1}e_{i,i+1}$ where $d_1 = 1$, $d_i = \prod_{m=2}^{i} a_{i-m+1,i-m+2}^{(i-m+1)}$, $i = 2, \ldots, n + 1$. Applying $\lambda_d\omega\varphi$ to $e_{i,i+1}$ we get the result. Let $\omega = g$ when $a_{n,n+1}^{(1)} \in R^*$. Then applying $\omega\varphi$ to $e_{i,i+1}$ we have that $\omega\varphi(e_{i,i+1}) = a_{i,i+1}^{(i)}e_{i,i+1} + t_2$, where $a_{i,i+1}^{(i)} \in R^*$, $i = 1, 2, \ldots, n$. Again, applying λ_d to $\omega\varphi(e_{i,i+1})$, we obtain that $\lambda_d\omega\varphi(e_{i,i+1}) = e_{i,i+1} + t_2$, $i = 1, 2, \ldots, n$. $\square$

Lemma 2.5. *Suppose that $n \geqslant 3$ and φ is in $\mathrm{Aut}(\mathbf{n}_1)$ such that $\varphi(e_{i,i+1}) = e_{i,i+1} + t_2$. Then*

$$\varphi(e_{12}) = e_{12} + a_{13}^{(1)}e_{13} + t_3,$$

$$\varphi(e_{i,i+1}) = e_{i,i+1} + a_{i-1,i+1}^{(i)}e_{i-1,i+1} + a_{i,i+2}^{(i)}e_{i,i+2} + t_3, \quad i = 2, \ldots, n-1, \tag{2.2}$$

$$\varphi(e_{n,n+1}) = e_{n,n+1} + a_{n-1,n+1}^{(n)}e_{n-1,n+1} + t_3,$$

where $a_{i+1,i+3}^{(i+2)} + a_{i,i+2}^{(i)} = 0$, $i = 1, \ldots, n-2$, and $t_3 \in \mathbf{n}_3$.

Proof. By the assumption we may write $\varphi(e_{i,i+1}) = e_{i,i+1} + \sum_{k=1}^{n-1} a_{k,k+2}^{(i)}e_{k,k+2} + t_3$, $i = 1, 2, \ldots, n$. In details,

$$\varphi(e_{12}) = e_{12} + a_{13}^{(1)}e_{13} + \sum_{k=2}^{n-1} a_{k,k+2}^{(1)}e_{k,k+2} + t_3,$$

$$\varphi(e_{23}) = e_{23} + a_{13}^{(2)} e_{13} + a_{24}^{(2)} e_{24} + \sum_{k=3}^{n-1} a_{k,k+2}^{(2)} e_{k,k+2} + t_3,$$

$$\begin{aligned}\varphi(e_{i,i+1}) &= e_{i,i+1} + \sum_{k=1}^{i-2} a_{k,k+2}^{(i)} e_{k,k+2} + a_{i-1,i+1}^{(i)} e_{i-1,i+1} + a_{i,i+2}^{(i)} e_{i,i+2} \\ &\quad + \sum_{k=i+1}^{n-1} a_{k,k+2}^{(i)} e_{k,k+2} + t_3, \quad i = 3, \ldots, n-2, \quad \text{here } n \geqslant 5,\end{aligned}$$

$$\begin{aligned}\varphi(e_{n-1,n}) &= e_{n-1,n} + \sum_{k=1}^{n-3} a_{k,k+2}^{(n-1)} e_{k,k+2} + a_{n-2,n}^{(n-1)} e_{n-1,n} \\ &\quad + a_{n-1,n+1}^{(n-1)} e_{n-1,n+1} + t_3, \quad \text{here } n \geqslant 4,\end{aligned}$$

$$\varphi(e_{n,n+1}) = e_{n,n+1} + \sum_{k=1}^{n-2} a_{k,k+2}^{(n)} e_{k,k+2} + a_{n-1,n+1}^{(n)} e_{n-1,n+1} + t_3.$$

By $[\varphi(e_{i,i+1})]^2 = 0$, $i = 1, \ldots, n$, we have $a_{i+1,i+3}^{(i)} = 0$ for $i = 1, \ldots, n-2$, and $a_{i-2,i}^{(i)} = 0$ for $i = 3, \ldots, n$. Now we distinguish the following four cases:

(i) From $[\varphi(e_{34}) \circ \varphi(e_{45})] \circ \varphi(e_{45}) = 0$, we have $a_{13}^{(4)} = 0$ (when $n \geqslant 4$). Then by $\varphi(e_{34}) \circ \varphi(e_{i,i+1}) = 0$, here $5 \leqslant i \leqslant n$, we have $a_{13}^{(i)} = 0$ for $5 \leqslant i \leqslant n$.
(ii) $2 \leqslant k \leqslant i-3 (i \geqslant 5)$. Since $\varphi(e_{k-1,k}) \circ \varphi(e_{i,i+1}) = 0$, we have $a_{k,k+2}^{(i)} = 0$ for $i = 5, \ldots, n$.
(iii) $i+2 \leqslant k \leqslant n-2$. By $\varphi(e_{i,i+1}) \circ \varphi(e_{k+2,k+3}) = 0$, we obtain $a_{k,k+2}^{(i)} = 0$ for $i = 1, \ldots, n-4$, here $n \geqslant 5$.
(iv) $k = n-1$. From the two equalities: $\varphi(e_{i,i+1}) \circ \varphi(e_{n-2,n-1}) = 0$ where $i = 1, \ldots, n-4$ ($n \geqslant 5$), and $\varphi(e_{n-1,n}) \circ [\varphi(e_{n-1,n}) \circ \varphi(e_{n,n+1})] = 0$ where $n \geqslant 4$, we obtain that $a_{n-1,n+1}^{(i)} = 0$ for $i = 1, \ldots, n-4$, and $a_{n-1,n+1}^{(n-3)} = 0$ respectively.

Summarizing the above results, we have

$$\varphi(e_{12}) = e_{12} + a_{13}^{(1)} e_{13} + t_3,$$

$$\varphi(e_{i,i+1}) = e_{i,i+1} + a_{i-1,i+1}^{(i)} e_{i-1,i+1} + a_{i,i+2}^{(i)} e_{i,i+2} + t_3, \quad i = 2, \ldots, n-1,$$

$$\varphi(e_{n,n+1}) = e_{n,n+1} + a_{n-1,n+1}^{(n)} e_{n-1,n+1} + t_3.$$

Furthermore, we have $a_{i+1,i+3}^{(i+2)} + a_{i,i+2}^{(i)} = 0$ from $\varphi(e_{i,i+1}) \circ \varphi(e_{i+2,i+3}) = 0$ where $i = 1, \ldots, n-2$. □

3. Lemmas for main results

Lemma 3.1. *Suppose $n \geqslant 3$. If $\varphi \in \mathrm{Aut}(\mathbf{n}_1)$ has the property: $\varphi(e_{i,i+1}) = e_{i,i+1} + t_2$, $i = 1, 2, \ldots, n$, then we may find an inner automorphism*

$$\theta = \theta_{r_{12}(-a_{13}^{(2)})} \prod_{i=1}^{n-2} \theta_{r_{i+1,i+2}(a_{i,i+2}^{(i)})} \theta_{r_{n,n+1}(a_{n-1,n+1}^{(n-1)})} \tag{3.1}$$

such that

$$\theta\varphi(e_{i,i+1}) = e_{i,i+1} + t_3, \quad i = 1, 2, \ldots, n.$$

Proof. By Lemma 2.5, $\varphi(e_{12})$, $\varphi(e_{i,i+1})$ $(i = 2, \ldots, n-1)$ and $\varphi(e_{n,n+1})$ can be written as the forms in (2.2). Note that $\theta_{r_{i+1,i+2}(a_{i,i+2}^{(i)})}(e_{i,i+1}) = e_{i,i+1} - a_{i,i+2}^{(i)} e_{i,i+2}$, $\theta_{r_{i+1,i+2}(a_{i,i+2}^{(i)})}(e_{i+2,i+3}) = e_{i+2,i+3} - a_{i+1,i+3}^{(i+2)} e_{i+1,i+3}$ $\quad (a_{i,i+2}^{(i)} = -a_{i+1,i+3}^{(i+2)}$, see Lemma 2.5), and $\theta_{r_{i+1,i+2}(a_{i,i+2}^{(i)})}(e_{kl}) = e_{kl} + t_3'$ (where $k \neq i+2, l \neq i+1$ and $l - k = 2$). So it is not difficult to show the conclusion. $\square$

Lemma 3.2. *Suppose $n \geqslant 3$ and $2 \leqslant m \leqslant [\frac{n+1}{2}]$ (the integer part of $\frac{n+1}{2}$). Let φ be in Aut$(\mathbf{n}_1)$ such that $\varphi(e_{i,i+1}) = e_{i,i+1} + t_m$. Then*

$$\varphi(e_{i,i+1}) = e_{i,i+1} + a_{i,i+m}^{(i)} e_{i,i+m} + t_{m+1} \quad \text{for } 1 \leqslant i \leqslant m-1,$$

$$\begin{aligned}\varphi(e_{i,i+1}) = {} & e_{i,i+1} + a_{i-m+1,i+1}^{(i)} e_{i-m+1,i+1} \\ & + a_{i,i+m}^{(i)} e_{i,i+m} + t_{m+1} \quad \text{for } m \leqslant i \leqslant n-m+1,\end{aligned} \tag{3.2}$$

$$\varphi(e_{i,i+1}) = e_{i,i+1} + a_{i-m+1,i+1}^{(i)} e_{i-m+1,i+1} + t_{m+1} \quad \text{for } n-m+2 \leqslant i \leqslant n,$$

where $a_{i+1,i+m+1}^{(i+m)} + a_{i,i+m}^{(i)} = 0$, $i = 1, 2, \ldots, n-m$.

Proof. When $m = 2$ that is the case in Lemma 2.5. And repeating the process of proving Lemma 2.5 we may verify the consequence. $\square$

Lemma 3.3. *Suppose that $n \geqslant 4$ and $[\frac{n+1}{2}] + 1 \leqslant m \leqslant n-1$. Let φ be in* Aut$(\mathbf{n}_1)$ *such that $\varphi(e_{i,i+1}) = e_{i,i+1} + t_m$. Then*

$$\varphi(e_{i,i+1}) = e_{i,i+1} + a_{i,i+m}^{(i)} e_{i,i+m} + t_{m+1} \quad \text{for } 1 \leqslant i \leqslant n-m+1,$$

$$\varphi(e_{i,i+1}) = e_{i,i+1} + t_{m+1} \quad \text{for } n-m+2 \leqslant i \leqslant m-1 \quad \left(m \neq \frac{n}{2} + 1, \text{ when } n \text{ even}\right),$$

$$\varphi(e_{i,i+1}) = e_{i,i+1} + a_{i-m+1,i+1}^{(i)} e_{i-m+1,i+1} + t_{m+1} \quad \text{for } m \leqslant i \leqslant n, \tag{3.3}$$

where $a_{i,i+m}^{(i)} + a_{i+1,i+m+1}^{(i+m)} = 0$, $i = 1, \ldots, n-m$.

Proof. The process for verifying the result is similar to that of Lemma 2.5. □

Lemma 3.4. *Suppose $n \geqslant 3$. If $\varphi \in \mathrm{Aut}(\mathbf{n}_1)$ has the property: $\varphi(e_{i,i+1}) = e_{i,i+1} + t_m$, $i = 1, 2, \ldots, n$, then we may find an inner automorphism*

$$\theta = \theta_{r_{1m}(-a^{(m)}_{1,m+1})} \prod_{i=1}^{n-m} \theta_{r_{i+1,i+m}(a^{(i)}_{i,i+m})} \theta_{r_{n-m+2,n+1}(a^{(n-m+1)}_{n-m+1,n+1})}, \quad 2 \leqslant m \leqslant n-1,$$

such that

$$\theta\varphi(e_{i,i+1}) = e_{i,i+1} + t_{m+1}, \quad i = 1, 2, \ldots, n.$$

Proof. Referring to the proof of Lemma 3.1 and applying the results of Lemma 3.2 and Lemma 3.3 to show that. □

Lemma 3.5. *Suppose $n \geqslant 3$. Let φ be in $\mathrm{Aut}(\mathbf{n}_1)$ such that $\varphi(e_{i,i+1}) = e_{i,i+1} + a^{(i)}_{1,n+1} e_{1,n+1}$, $i = 1, 2, \ldots, n$. Then we may find an inner automorphism*

$$\theta = \theta_{r_{2,n+1}(a^{(1)}_{1,n+1})} \theta_{r_{1n}(-a^{(n)}_{1,n+1})}$$

and a central automorphism ζ_{-c} with $c = \left(a^{(2)}_{1,n+1}, \ldots, a^{(n-1)}_{1,n+1}\right)$ such that

$$\zeta_{-c}\theta\varphi(e_{i,i+1}) = e_{i,i+1}, \quad i = 1, 2, \ldots, n.$$

Proof. Applying $\zeta_{-c}\theta$ to $\varphi(e_{i,i+1})$, we may find the result correct. □

4. Main results

Theorem 1. *Let R be a 2-torsionfree local ring with 1, and $N_{n+1}(R)$, $n \geqslant 3$, the algebra consisting of all strictly upper triangular $(n+1) \times (n+1)$ matrices over R. For any Jordan automorphism φ of $N_{n+1}(R)$ there are graph, diagonal, central and inner automorphisms, ω, λ_d, ζ_c and θ, respectively, of $N_{n+1}(R)$ such that $\varphi = \omega\lambda_d\zeta_c\theta$, and the decomposition $\varphi = \omega\lambda_d\zeta_c\theta$ is unique.*

Proof. By Lemmas 2.3–2.5 and Lemmas 3.1–3.5, we have

$$\zeta_{-c}\theta^{-1}\lambda_d^{-1}\omega\varphi(e_{i,i+1}) = e_{i,i+1}, \quad i = 1, 2, \ldots, n.$$

Since $e_{i,i+1}$, $i = 1, 2, \ldots, n$, generate $N_{n+1}(R)$, so

$$\varphi = \omega\lambda_d\theta\zeta_c = \omega\lambda_d\zeta_c\theta'.$$

The uniqueness of the decomposition follows from the following lemma. □

Lemma 4.1. *Let $\mathscr{G}$, $\mathscr{D}$, $\mathscr{C}$ and $\mathscr{I}$ be the graph, diagonal, central and inner automorphism group respectively. When $n \geqslant 3$, then*

$$\mathrm{Aut}(\mathbf{n}_1) = \mathscr{G} \ltimes (\mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I})).$$

Proof. Since $\mathscr{I} \lhd \mathrm{Aut}(\mathbf{n}_1)$, the product $\mathscr{C}\mathscr{I}$ is a group. Apparently $\mathscr{C} \cap \mathscr{I} = 1$ (the identity of $\mathrm{Aut}(\mathbf{n}_1)$), so $\mathscr{C}\mathscr{I} = \mathscr{C} \ltimes \mathscr{I}$. It is easy to see that the diagonal automorphism group $\mathscr{D}$ normalizes $\mathscr{C}$ and $\mathscr{I}$, then $\mathscr{D}(\mathscr{C} \ltimes \mathscr{I})$ is a subgroup of $\mathrm{Aut}(\mathbf{n}_1)$. Since any automorphism in $\mathscr{C} \ltimes \mathscr{I}$ acts trivially on $e_{i,i+1}$ mod $\mathbf{n}_2$, $i = 1, 2, \ldots, n$. Hence $\mathscr{D} \cap (\mathscr{C} \ltimes \mathscr{I}) = 1$, this means $\mathscr{D}(\mathscr{C} \ltimes \mathscr{I}) = \mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I})$. Finally, since the graph automorphism ω normalizes each of the subgroups $\mathscr{D}$, $\mathscr{C}$ and $\mathscr{I}$, $\mathscr{G}(\mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I}))$ is a subgroup of $\mathrm{Aut}(\mathbf{n}_1)$. Moreover, $\omega(e_{n,n+1}) \notin Re_{n,n+1}$ mod $\mathbf{n}_2$. But for any $\eta \in \mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I})$, $\eta(e_{n,n+1}) \in Re_{n,n+1}$ mod $\mathbf{n}_2$. Hence $\mathscr{G} \cap (\mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I})) = 1$, and $\mathscr{G}(\mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I})) = \mathscr{G} \ltimes (\mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I}))$.

Therefore, by the first part of Theorem 4.1, $\mathrm{Aut}(\mathbf{n}_1) = \mathscr{G} \ltimes (\mathscr{D} \ltimes (\mathscr{C} \ltimes \mathscr{I}))$. □

Discussion for $n = 1, 2$

When $n = 1$, it is obvious that $\varphi(e_{12}) = ae_{12}$, $a \in R^*$. Take $d = e_{11} + ae_{22}$, then $\lambda_d \varphi(e_{12}) = e_{12}$. So $\varphi = \lambda_{d^{-1}}$.

When $n = 2$ we have $\lambda_d \omega \varphi(e_{12}) = e_{12} + a_{13}^{(1)} e_{13}$ and $\lambda_d \omega \varphi(e_{23}) = e_{23} + a_{13}^{(2)} e_{13}$ by Lemma 2.4. Further, $\theta_{r_{12}(-a_{13}^{(2)})} \theta_{r_{23}(a_{13}^{(1)})} \lambda_d \omega \varphi(e_{12}) = e_{12}$ and $\theta_{r_{12}(-a_{13}^{(2)})} \theta_{r_{23}(a_{13}^{(1)})} \lambda_d \omega \varphi(e_{23}) = e_{23}$. Hence $\varphi = \omega \lambda_{d^{-1}} \theta_{r_{23}(-a_{13}^{(1)})} \theta_{r_{12}(a_{13}^{(2)})}$.

Note. Comparing the decomposition of Jordan automorphisms of $N_{n+1}(R)$ with that of Lie automorphisms of $N_{n+1}(R)$ (see [6]) under the same conditions on ring R, we may find that the extremal automorphism which is used in the decomposition of Lie automorphism of $N_{n+1}(R)$ is not needed here.

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